

NNLO corrections to 3-jet observables in electron-positron annihilation

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I report on a numerical program, which can be used to calculate any infrared safe three-jet observable in electron-positron annihilation to next-to-next-to-leading order in the strong coupling constant α_s . The results are compared to a recent calculation by another group. Numerical differences in three colour factors are discussed and explained.

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INTRODUCTION

Jet observables and event shapes in electron-positron annihilation can be used to extract the value of the strong coupling constant α_s [1, 2, 3]. This applies in particular to three-jet observables, where the leading-order parton process is proportional to α_s . In order to extract the numerical value from the LEP data, precise theoretical calculations are necessary, calling for a next-to-next-to-leading order (NNLO) calculation. Due to the large variety of interesting jet observables it is desirable not to perform this calculation for a specific observable, but to set up a computer program, which yields predictions for any infra-red safe observable relevant to the process $e^+e^- \rightarrow 3$ jets. Such a task requires the calculation of the relevant amplitudes up to two loops, a method for the cancellation of infrared divergences and stable and efficient Monte Carlo techniques. For the process $e^+e^- \rightarrow 2$ jets this was done in [4, 5, 6, 7]. In this letter I report on a NNLO calculation for three-jet observables in electron-positron annihilation. Recently another group published results for the NNLO corrections for three-jet observables [8, 9, 10, 11]. In the calculation presented here the methods used are in many parts similar to the ones used in [8, 9, 10, 11], although I will show that in certain points there are important differences. The authors of [8, 9, 10, 11] made major contributions to the development of these methods [5, 12, 13, 14, 15].

The numerical results of the two calculations are compared. The comparison is facilitated by splitting the NNLO correction term into individually gauge-invariant contributions, such that each contribution is proportional to a specific colour factor. For the NNLO corrections to $e^+e^- \rightarrow 3$ jets there are six different colour factors. In three colour factors the two calculations agree (N_c^{-2} , N_f/N_c , N_f^2). They disagree in the remaining three colour factors (N_c^2 , N_c^0 , $N_f N_c$). The numerical differences in these colour factors can be traced back to an incomplete cancellation of soft-gluon singularities in the calculation of refs. [8, 9, 10, 11]. These singularities require additional subtraction terms, which are subtracted from the five-parton configuration and added to the four-parton configuration. These subtraction terms have a structure

not present in [10] and are related to soft gluons. These terms occur generically in any NNLO calculation with three or more hard coloured partons.

GENERAL SET-UP

The perturbative expansion of any infrared-safe observable for the process $e^+e^- \rightarrow 3$ jets can be written up to NNLO as

$$\mathcal{O} = \frac{\alpha_s}{2\pi} A_{\mathcal{O}} + \left(\frac{\alpha_s}{2\pi}\right)^2 B_{\mathcal{O}} + \left(\frac{\alpha_s}{2\pi}\right)^3 C_{\mathcal{O}}. \quad (1)$$

$A_{\mathcal{O}}$ gives the LO result, $B_{\mathcal{O}}$ the NLO correction and $C_{\mathcal{O}}$ the NNLO correction. The coefficient $C_{\mathcal{O}}$ can be decomposed into colour pieces

$$C_{\mathcal{O}} = \frac{1}{8} (N_c^2 - 1) \left[N_c^2 C_{\mathcal{O}}^{lc} + C_{\mathcal{O}}^{sc} + \frac{1}{N_c^2} C_{\mathcal{O}}^{ssc} \right. \\ \left. + N_f N_c C_{\mathcal{O}}^{nf} + \frac{N_f}{N_c} C_{\mathcal{O}}^{nfs} + N_f^2 C_{\mathcal{O}}^{nfnf} \right], \quad (2)$$

where N_c denotes the number of colours and N_f the number of light quark flavours. In addition, there are singlet contributions, which arise from interference terms of amplitudes, where the electro-weak boson couples to two different fermion lines. These singlet contributions are expected to be numerically small [16, 17, 18] and neglected in the present calculation.

The computation of the NNLO coefficient $C_{\mathcal{O}}$ requires the knowledge of the amplitudes for the three-parton final state $e^+e^- \rightarrow \bar{q}qg$ up to two-loops [18, 19], the amplitudes of the four-parton final states $e^+e^- \rightarrow \bar{q}qgg$ and $e^+e^- \rightarrow \bar{q}q\bar{q}q$ up to one-loop [20, 21, 22, 23] and the five-parton final states $e^+e^- \rightarrow \bar{q}qggg$ and $e^+e^- \rightarrow \bar{q}q\bar{q}qg$ at tree level [24, 25]. Taken separately, the three-, four- and five-parton contributions are all individually infrared divergent. Only the sum of them is finite. However, the individual contributions live on different phase spaces, which prevents a naive Monte Carlo approach. To render the individual contributions finite, several options for the cancellation of infrared divergences have been discussed, like phase space slicing [26], sector decomposition [27, 28], a method based on the optical theorem [29] or

the subtraction method [5, 12, 13, 14, 15, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. In the present calculation I use the subtraction method with antenna subtraction terms [15].

CANCELLATION OF DIVERGENCES

To render the individual three-, four- and five-parton contributions finite, one adds and subtracts suitable chosen terms. Schematically, we have

$$\begin{aligned} 5 \text{ partons : } & d\sigma_5^{(0)} - d\alpha^{NLO} - d\alpha^{NNLO} + d\alpha^{iterated} \\ & - d\alpha^{almost} - d\alpha^{soft}, \\ 4 \text{ partons : } & d\sigma_4^{(1)} + d\alpha^{NLO} - d\alpha^{loop} - d\alpha^{iterated} \\ & - d\alpha^{product} + d\alpha^{almost} + d\alpha^{soft}, \\ 3 \text{ partons : } & d\sigma_3^{(2)} + d\alpha^{NNLO} + d\alpha^{loop} + d\alpha^{product}. \end{aligned}$$

Here, $d\sigma_5^{(0)}$, $d\sigma_4^{(1)}$ and $d\sigma_3^{(2)}$ are the contributions from the original amplitudes with five, four or three final state partons. $d\alpha^{NLO}$ is the NLO subtraction term for four-jet observables, containing only three parton tree-level antenna functions. At NNLO there are several new subtraction terms required, each of them with a specific structure. The term $d\alpha^{NNLO}$ contains the four-parton tree-level antenna functions. The term $d\alpha^{loop}$ contains three-parton one-loop antenna functions together with tree-level matrix elements and three-parton tree-level antenna functions together with one-loop matrix elements. The remaining terms $d\alpha^{iterated}$, $d\alpha^{almost}$, $d\alpha^{product}$ and $d\alpha^{soft}$ all contain a product of two three-parton tree-level antenna functions. In $d\alpha^{iterated}$ and $d\alpha^{almost}$ one antenna function has five-parton kinematics, while the other antenna has four-parton kinematics. The former subtraction term is an approximation to $d\alpha^{NLO}$, while the latter approximates $d\sigma_5^{(0)}$ in almost colour-correlated double unresolved configurations. In $d\alpha^{product}$ both antennas have four-parton kinematics. The term $d\alpha^{soft}$ will be discussed below and is relevant only for the colour factors N_c^2 , N_c^0 and $N_f N_c$.

The subtraction terms without $d\alpha^{soft}$ correspond to the subtraction scheme of ref. [10]. For any subtraction scheme it is required, that in the three-parton channel the explicit divergences cancel, that the four-parton channel is integrable over a single unresolved phase space and in addition that the explicit divergences cancel and finally that in the five-parton channel the integrand is integrable over single and double unresolved phase space regions. It is easily checked that with the subtraction terms of ref. [10] the explicit divergences in the three-parton cancel and I will focus in the following on the four- and five-parton channels.

In the four-parton channel the combination $d\sigma_4^{(1)} + d\alpha^{NLO}$ is free of explicit poles. It has been noted in ref. [10] that the combination $d\alpha^{loop} + d\alpha^{iterated} +$

$d\alpha^{product} - d\alpha^{almost}$ involves in the colour factors N_c^2 and N_c^0 poles of the form

$$\left| \mathcal{A}_3^{(0)}(1', 2', j) \right|^2 X_3^0(1, i, 2) \frac{1}{\epsilon} \left[\ln \frac{s_{1'j} s_{j2'}}{s_{1'2'}} - \ln \frac{s_{1j} s_{j2}}{s_{12}} \right],$$

where $p_{1'}$ and $p_{2'}$ are the momenta obtained from p_1 , p_i and p_2 through a $3 \rightarrow 2$ phase space map. $\mathcal{A}_3^{(0)}$ is the three-parton tree-level amplitude and $X_3^0(1, i, 2)$ a three-parton tree-level antenna function. In ref. [10] it was claimed that these poles vanish after the azimuthal integration over the unresolved phase space. This claim is wrong. In the centre-of-mass frame of $p_{1'} + p_{2'}$ with $p_{1'}$ and p_1 along the positive z -axis, the relevant integral is

$$I = \frac{1}{2\pi} \int_0^{2\pi} d\phi \ln \left(\frac{(1 + c_j)(1 - c_2)}{2(1 - c_2 c_j - s_2 s_j \cos \phi)} \right), \quad (3)$$

where for $x = 2, j$ we set $c_x = \cos \theta_x$, $s_x = \sin \theta_x$ and θ_2 and θ_j are the polar angles of partons 2 and j in the chosen frame. The integral equals

$$I = \ln \left(\frac{1 - c_2 c_j + (c_j - c_2)}{1 - c_2 c_j + |c_j - c_2|} \right). \quad (4)$$

The integral is zero for $\theta_j < \theta_2$ but non-zero for $\theta_j > \theta_2$. In ref. [10] it was claimed that the integral vanishes in both cases. As a consequence of the non-zero value for $\theta_j > \theta_2$ the explicit poles do not cancel in the combination $d\alpha^{loop} + d\alpha^{iterated} + d\alpha^{product} - d\alpha^{almost}$. The same situation occurs also in the colour factor $N_f N_c$.

These poles have a counter-part in the five-parton channel. Setting $d\alpha^{soft}$ to zero and using a slicing approach one observes in the colour factors N_c^2 , N_c^0 and $N_f N_c$ a logarithmic dependence on the slicing parameter $y_{min} = s_{min}/Q^2$. This is shown in fig. 1.

These singularities require an additional subtraction term and that is where the present calculation differs from the one of ref. [8, 9, 10, 11]. $d\alpha^{soft}$ is a subtraction term related to soft gluons which ensures that the poles in the four-parton configuration vanish after integration over the azimuthal angle and which renders the five-parton configuration independent of y_{min} . The term $d\alpha^{soft}$ for the four-parton configuration can be taken of the form

$$\left| \mathcal{A}_3^{(0)}(1', 2', j) \right|^2 X_3^0(1, i, 2) \theta \left(\frac{2p_1 p_j}{2p_{1i2} p_j} - \frac{2p_1 p_2}{2p_{1i2} p_2} \right) \left[\mathcal{S}_3^0(s_{1j}) - \mathcal{S}_3^0(s_{12}) - \mathcal{S}_3^0(s_{2j}) + \mathcal{S}_3^0(s_{22}) \right], \quad (5)$$

where \mathcal{S}_3^0 is the integrated soft antenna function and p_2 is given by

$$p_2 = p_2 + p_i - \frac{s_{2i}}{s_{12} + s_{1i}} p_1. \quad (6)$$

y_{cut}	A_{3-jet}	B_{3-jet}	C_{3-jet}
0.3	0.02	0.13	-6 ± 3
0.1	2.12	34.3	$(2.0 \pm 0.2) \cdot 10^2$
0.03	7.63	113.8	$(6.7 \pm 0.6) \cdot 10^2$
0.01	15.7	152.6	$(-1.2 \pm 0.2) \cdot 10^3$
0.003	27.9	-6.5	$(-8.1 \pm 0.5) \cdot 10^3$
0.001	42.4	-562	$(-21 \pm 1) \cdot 10^3$
0.0003	61.8	$-1.97 \cdot 10^3$	$(-25 \pm 3) \cdot 10^3$
0.0001	82.9	$-4.36 \cdot 10^3$	$(7 \pm 5) \cdot 10^3$

TABLE I: The LO coefficient A_{3-jet} , the NLO coefficient B_{3-jet} and the NNLO coefficient C_{3-jet} for the three jet cross section with the Durham jet algorithm and various values of y_{cut} .

I also used the short-hand notation $p_{1i2} = p_1 + p_i + p_2$. The θ -function enforces $\theta_j > \theta_2$ in the specific frame introduced above. $d\alpha^{soft}$ for the five-parton configuration is obtained by lifting eq. 5 to the five parton phase space. Fig. 1 shows that the sum of all contributions in the five-parton channel is now independent of y_{min} . I have checked that in the four-parton channel the explicit poles cancel after integration over the unresolved phase space.

NUMERICAL RESULTS

The numerical program is build on an existing NLO program for $e^+e^- \rightarrow 4$ jets [42]. I consider the three-jet cross section, where the jets are defined by the Durham jet algorithm [43]. The recombination prescription is given by the E-scheme. I take the centre of mass energy to be $\sqrt{Q^2} = m_Z$. The three-jet cross section is expanded as

$$\sigma_{3-jet} = \sigma_0 \left[\frac{\alpha_s}{2\pi} A_{3-jet} + \left(\frac{\alpha_s}{2\pi} \right)^2 B_{3-jet} + \left(\frac{\alpha_s}{2\pi} \right)^3 C_{3-jet} \right],$$

where σ_0 is the LO cross section for $e^+e^- \rightarrow$ hadrons. The coefficients A_{3-jet} , B_{3-jet} and C_{3-jet} are given for the renormalisation scale $\mu^2 = Q^2$ and various values of the jet defining parameter y_{cut} in table I. The errors of C_{3-jet} are from the Monte Carlo integration. For selected values of y_{cut} the contribution from the individual colour factors to the NNLO coefficient C_{3-jet} is shown in table II. Finally, fig. 2 shows the scale variation of the jet rate defined by

$$\frac{\sigma_{3-jet}}{\sigma_{tot}} = \frac{\alpha_s}{2\pi} \bar{A}_{3-jet} + \left(\frac{\alpha_s}{2\pi} \right)^2 \bar{B}_{3-jet} + \left(\frac{\alpha_s}{2\pi} \right)^3 \bar{C}_{3-jet},$$

where

$$\begin{aligned} \bar{A}_{3-jet} &= A_{3-jet}, \quad \bar{B}_{3-jet} = B_{3-jet} - A_{3-jet} A_{tot}, \\ \bar{C}_{3-jet} &= C_{3-jet} - B_{3-jet} A_{tot} - A_{3-jet} (B_{tot} - A_{tot}^2) \end{aligned}$$

y_{cut}	N_c^2	$N_f N_c$
0.1	$(1.06 \pm 0.02) \cdot 10^3$	$(-9.80 \pm 0.06) \cdot 10^2$
0.01	$(4.6 \pm 0.2) \cdot 10^3$	$(-8.11 \pm 0.03) \cdot 10^3$
0.001	$(-29 \pm 1) \cdot 10^3$	$(-2.7 \pm 0.2) \cdot 10^3$
y_{cut}	N_c^0	N_f/N_c
0.1	-35 ± 1	21.9 ± 0.3
0.01	$(9.7 \pm 0.1) \cdot 10^2$	$(-2.66 \pm 0.02) \cdot 10^2$
0.001	$(7.09 \pm 0.08) \cdot 10^3$	$(-4.43 \pm 0.01) \cdot 10^3$
y_{cut}	N_c^{-2}	N_f^2
0.1	-0.49 ± 0.03	$(1.336 \pm 0.003) \cdot 10^2$
0.01	0.25 ± 0.15	$(1.646 \pm 0.002) \cdot 10^3$
0.001	$(3.38 \pm 0.01) \cdot 10^2$	$(7.41 \pm 0.01) \cdot 10^3$

TABLE II: The contributions from the individual colour factors to the NNLO coefficient C_{3-jet} .

and $A_{tot} = 2$,

$$B_{tot} = \frac{N_c^2 - 1}{8N_c} \left[\left(\frac{243}{4} - 44\zeta_3 \right) N_c + \frac{3}{4N_c} + (8\zeta_3 - 11) N_f \right].$$

The renormalisation scale is varied from $\mu = m_Z/2$ to $\mu = 2m_Z$. In this plot the experimental measured values are also shown [44]. For values below $y_{cut} = 0.001$ the results of ref. [8] differ significantly from the ones presented here.

CONCLUSIONS

In this letter I reported on the NNLO calculation for three-jet observable in electron-positron annihilation. Particular attention was paid to the cancellation of infrared singularities. I presented numerical results for the Durham three-jet cross section.

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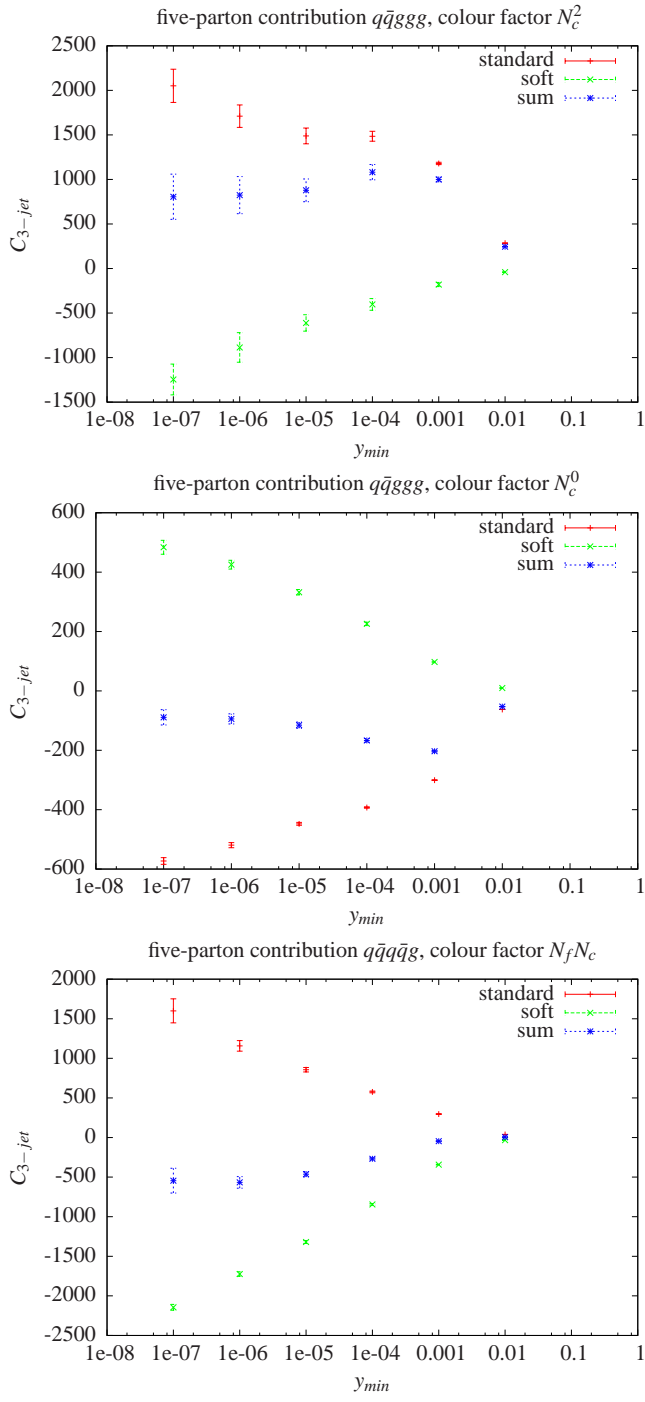


FIG. 1: Dependence of the five-parton contribution on the slicing parameter y_{min} for the Durham jet cross section with $y_{cut} = 0.01$ in the colour factors N_c^2 , N_c^0 and $N_f N_c$. “Standard” denotes the combination $d\sigma_5^{(0)} - d\alpha^{NLO} - d\alpha^{NNLO} + d\alpha^{iterated} - d\alpha^{almost}$, “soft” the contribution from $d\alpha^{soft}$. In addition the sum of the two terms is shown. For small values of y_{min} the sum is independent of y_{min} .

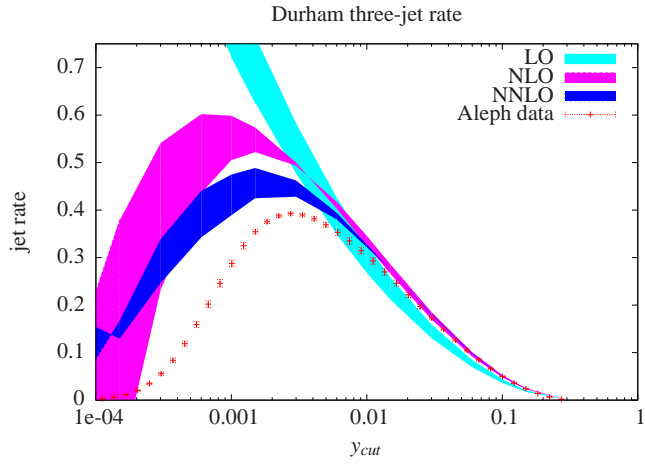


FIG. 2: The scale variation of the three jet rate with the Durham jet algorithm at $\sqrt{Q^2} = m_Z$ with $\alpha_s(m_Z) = 0.118$. The bands give the range for the theoretical prediction obtained from varying the renormalisation scale from $\mu = m_Z/2$ to $\mu = 2m_Z$.